Environment-assisted creation and enhancement of two-mode entanglement in a joint decoherent model

Shao-Hua Xiang,1,2,3 Bin Shao,1 and Ke-Hui Song2,3

1Department of Physics, Beijing Institute of Technology, Beijing 100081, People’s Republic of China
2Research Institute of Information Science, Huaihua University, Huaihua 418008, People’s Republic of China
3Department of Physics and Electronic Information Science, Huaihua University, Huaihua, Hunan 418008, People’s Republic of China

(Received 21 June 2008; published 10 November 2008)

We investigate the entanglement dynamics of two modes in a common squeezed thermal environment by means of the characteristic function method and the logarithmic negativity. When the two modes are prepared initially in separable squeezed states, we find that the external environment can not only induce but also enhance the two-mode entanglement if their initial squeezing parameters are suitably chosen. When the two modes are initially in a two-mode squeezed vacuum state, we find that whether the initial entanglement is degraded or increased depends strongly on the difference between the squeezing parameters of the considered bath and of a quantum state. Finally, we give a condition of all two-mode entangled Gaussian states to be the decoherence-free state in a given squeezed bath.

DOI: 10.1103/PhysRevA.78.052313 PACS number(s): 03.67.Mn, 03.67.Hk, 03.65.Ud

I. INTRODUCTION

A two-mode Gaussian state is the simplest nontrivial instance of multipartite entangled Gaussian states and has attracted much interest in recent years, as it plays a crucial role in quantum-information processing with continuous variables of multipartite entangled Gaussian states and has attracted much interest in recent years, as it plays a crucial role in quantum-information processing with continuous variables (CV), such as quantum teleportation [1], quantum cryptography [2], quantum secure communication [3], and quantum entanglement swapping [4]. Thus, a considerable effort has been devoted to generating two-mode Gaussian entanglement in a variety of physical situations, e.g., optical parametric amplifiers [5], Kerr nonlinearity in an optical fiber [6] or cavities [7], and two-mode cavity quantum electrodynamics (QED) [8]. In particular, two-mode squeezed vacuum states have been generated experimentally via the nonlinear process of parametric amplifiers [9] and the combination of single-mode squeezing and beam splitting [6], respectively.

Any attempt to implement quantum communication and computation has to face the obvious difficulty that a realistic quantum-mechanical system is never isolated and inevitably interacts with its surrounding environment, which contains a large number of uncontrollable degrees of freedom. It is often argued that this unwanted interaction can lead to the degradation of purity, distinguishability, and entanglement of quantum states [10]. As a consequence, the entanglement-based task will be accomplished only partially and even fail completely. This deleterious effect is usually called decoherence. But this assertion turns out not to be always true. Recently, notable examples of environment-assisted entanglement creation and enhancement have been proposed. For instance, it was found that two qubits without a direct coupling can be entangled by the interaction via a common single-mode [11,12] or two-mode [13] thermal environment or via a common environment with a gap [14]. In addition, Ghosh et al. [15] have investigated concrete cases of the increase of entanglement caused by the stronger interaction of a part of a composite system with its environment. Note that there exist some special kinds of entangled states that are immune to a number of noise influences [16]. Such states are termed coherence-preserving states or decoherence-free states (DFS). It has been shown that the usage of DFS is an effective way to bypass decoherence in quantum-information processing. But, to our knowledge, few attempts to study the above-mentioned interesting issues in the CV systems have been made so far.

Investigations of the decoherence and loss of two-mode Gaussian states under open system dynamics have recently received much attention. In those investigations, the independent decoherence models are considered, that is, two modes are assumed to be spatially separated and independently coupled to its own thermal bath [17] or squeezed bath [18]. It has been shown that two-mode Gaussian entanglement is rather fragile and easily destroyed by the external environment. For example, Wilson et al. [18] investigated the entanglement properties of a two-mode squeezed state in a phase-sensitive Gaussian environment using by the Weyl characteristic method, in which the interaction of a field with the external environment can be described by the field passing through an array composed of an infinite number of beam splitters. The physical properties of the environment are determined by the fields injected into unused ports of the beam splitters. It has been found that their approach is quite powerful for dealing with the dynamics of the continuous-variable systems in independent noise, but it cannot be extended to the case of many field modes embedded in the common environment. On the other hand, decoherence and entanglement degradation of such states in a common environment have been investigated in the past few years. Prazuuner [19] investigated the entanglement dynamics of a two-mode squeezed vacuum state in a common environment and showed that if such a state is initially sufficiently squeezed, it will always remain entangled regardless of the strength of the interaction to the external bath. Horhammer et al. [20] studied the dynamics of two-mode squeezed states in a common bath within an extended quantum Brownian mo-
tion model and showed that below a critical bath temperature, two-mode entanglement is preserved even in the steady state. paz and roncaglia [21] also analyzed the entanglement properties of two oscillators in the same environment by means of the exact master equation for quantum brownian motion and showed that the entanglement can undergo three phases: sudden death, sudden death and revival, and no sudden death. however, in previous works, the common bath is assumed to be a thermal environment with initial temperature, which can result in the loss of the initial entanglement. a natural and important question then arises: could the noise environment induce and increase the amount of entanglement between the subsystems of CV systems? does there exist a CV version of DFS in a noise environment?

To this aim, we consider a joint decoherent model, in which the two noninteracting modes are coupled to a common squeezed thermal bath. we first derive the Fokker-Planck equation for this model and give its general solution associated with a generic two-mode Gaussian state. subsequently, we investigate the entanglement dynamics of the two modes initially in various Gaussian states—single-mode squeezed vacuum and two-mode squeezed vacuum states—by means of the logarithmic negativity. we establish the conditions of creating entanglement between two independent squeezed states and of a CV version of the decoherence-free states in our joint decoherent model. Finally, we address an interesting issue, namely the relation between whether entanglement increases or decreases and the characteristic parameter of the squeezed bath.

II. GENERAL SOLUTION TO THE MASTER EQUATION

Let us consider a joint decoherent model, in which two noninteracting harmonic oscillators are immersed in a common noisy environment. The Hilbert space of the whole system is a tensor product of the subsystems’ Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Each subsystem is described by dimensionless quadrature operators defined as $\hat{x}_i = (\hat{a}_i^+ + \hat{a}_i) / \sqrt{2}$ and $\hat{p}_i = i(\hat{a}_i^+ - \hat{a}_i) / \sqrt{2}$. These operators obey a standard commutation relation $[\hat{x}_i, \hat{p}_j] = i \hbar \delta_{ij}$, which is equivalent to $[\hat{a}_i, \hat{a}^\dagger_j] = i \delta_{ij}$. We assume that the two harmonic oscillators have the same frequency, $\omega_1 = \omega_2 = \omega$, and should act homogeneously with the environment. The Hamiltonian in the rotating-wave approximation describing the two harmonic oscillators coupled to the same radiation bath is thus given by

$$\hat{H} = \hbar \sum_{i=1,2} \omega \hat{a}^\dagger_i \hat{a}_i + \hbar \sum_{i=1,2} (\hat{\Gamma} \hat{a}_i^\dagger + \hat{\Gamma}^\dagger \hat{a}_i) + \hat{H}_{\text{bath}}, \quad (1)$$

where $\hat{H}_{\text{bath}}$ indicates the free Hamiltonian of the external environment, $\hat{\Gamma} = \sum_{i=1,2} \hat{g}_i \hat{b}_i$ and $\hat{\Gamma}^\dagger = \sum_{i=1,2} \hat{g}^\ast_i \hat{b}_i^\dagger$, with the operators $\hat{b}_i$ and $\hat{b}_i^\dagger$ being modes of the reservoir that damp the field with index $k$, and $\hat{g}_k$ being the coupling between the environment and the system.

Without loss of generality, the reservoir is assumed to be a squeezed environment with the following correlations [22]:

$$\langle \hat{\Gamma}^\dagger(t) \hat{\Gamma}(t') \rangle = \gamma N \delta(t - t'), \quad (2a)$$

where $\gamma$ is the field-decay rate, $N$ represents the mean photon number of the squeezed reservoir, and $M$ is a parameter related to the phase correlations of the squeezed reservoir. The Heisenberg uncertainty relation imposes the constraint $|M|^2 = N(N+1)$.

In the interaction picture and Markovian approximation, we can obtain the master equation for the reduced density matrix of the system as

$$\frac{\partial}{\partial t} \rho = \frac{\gamma}{2} \sum_{i=1,2} (N + 1) (2 \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i - \hat{a}_i^\dagger \hat{a}_i \rho - \rho \hat{a}_i \hat{a}_i^\dagger) + N(2 \hat{a}_i^\dagger \hat{p}_i - \hat{a}_i^\dagger \hat{a}_i \rho - \rho \hat{a}_i \hat{a}_i^\dagger) + M(2 \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i - \hat{a}_i^\dagger \hat{a}_i \rho - \rho \hat{a}_i \hat{a}_i^\dagger) + M^*(2 \hat{a}_i \hat{p}_i - \hat{a}_i \hat{a}_i \rho - \rho \hat{a}_i \hat{a}_i) + \sum_{j \neq i=1,2} (N + 1) (2 \hat{a}_i^\dagger \hat{p}_i^\dagger - \hat{a}_j^\dagger \hat{a}_j^\dagger \rho - \rho \hat{a}_j^\dagger \hat{a}_j^\dagger) + M(2 \hat{a}_i \hat{p}_i - \hat{a}_i \hat{a}_i \rho - \rho \hat{a}_i \hat{a}_i) + M^*(2 \hat{a}_i \hat{p}_i - \hat{a}_i \hat{a}_i \rho - \rho \hat{a}_i \hat{a}_i) \right). \quad (3)$$

It should be pointed out that the first four terms in Eq. (3) describe the individual dissipations of each mode due to the environment, while the other terms denote the couplings between the modes induced by the common bath. For the reservoir in a squeezed vacuum state, i.e., $|M|^2 = N(N+1)$, Eq. (3) reduces to a rather simple master equation with only one Lindblad operator,

$$\frac{\partial}{\partial t} \rho = \frac{\gamma}{2} (2 \hat{F} \rho \hat{F}^\dagger - \hat{F}^\dagger \hat{F} \rho - \rho \hat{F}^\dagger \hat{F}), \quad (4)$$

where the operator $\hat{F}$ is given by

$$\hat{F} = \hat{F}_1 + \hat{F}_2 = \sqrt{N+1} (\hat{a}_1^\dagger + \hat{a}_2^\dagger) + \sqrt{N} e^{i \theta} (\hat{a}_1^\dagger + \hat{a}_2^\dagger) \quad \text{with} \ \theta \ \text{being the reference phase for the squeezed field.} \quad (5)$$

Using the standard operator correspondence [23], we can transform Eq. (3) into the following Fokker-Planck equation for the characteristic function (CF):

$$\frac{\partial}{\partial t} \chi(\beta, t) = -\frac{\gamma}{2} \left[ \left(2N + 1 \right) |\beta_1|^2 + |\beta_2|^2 + |\beta_1^\dagger \beta_2^\dagger + \beta_1 \beta_2^\dagger + \beta_1^\dagger \beta_2 \right] + M \left[ |\beta_1|^2 + |\beta_2|^2 + 2 |\beta_1^\dagger \beta_2^\dagger | \right] + M^* \left[ |\beta_1^\dagger \beta_2^\dagger | + |\beta_1 \beta_2^\dagger | \right] \frac{\partial}{\partial \beta_1^\dagger} + \beta_1^\dagger \frac{\partial}{\partial \beta_1} + \beta_2^\dagger \frac{\partial}{\partial \beta_2} + \beta_1 \frac{\partial}{\partial \beta_1^\dagger} + \beta_2 \frac{\partial}{\partial \beta_2^\dagger}.$$

$\chi(\beta, t)$ being the reference phase for the squeezed field. It is not difficult to verify that the operators $\hat{F}_1$ and $\hat{F}_2$ satisfy the commutation relation $[\hat{F}_1, \hat{F}_2^\dagger] = i \delta_{ij}$. Using the standard operator correspondence [23], we can transform Eq. (3) into the following Fokker-Planck equation for the characteristic function (CF):

$$\frac{\partial}{\partial t} \chi(\beta, t) = -\frac{\gamma}{2} \left[ \left(2N + 1 \right) |\beta_1|^2 + |\beta_2|^2 + |\beta_1^\dagger \beta_2^\dagger + \beta_1 \beta_2^\dagger + \beta_1^\dagger \beta_2 \right] + M \left[ |\beta_1|^2 + |\beta_2|^2 + 2 |\beta_1^\dagger \beta_2^\dagger | \right] + M^* \left[ |\beta_1^\dagger \beta_2^\dagger | + |\beta_1 \beta_2^\dagger | \right] \frac{\partial}{\partial \beta_1^\dagger} + \beta_1^\dagger \frac{\partial}{\partial \beta_1} + \beta_2^\dagger \frac{\partial}{\partial \beta_2} + \beta_1 \frac{\partial}{\partial \beta_1^\dagger} + \beta_2 \frac{\partial}{\partial \beta_2^\dagger}.$$
\[ \chi(\beta,t) = \exp \left\{ -\frac{1}{4} [(2N + 1)\tilde{K} + M\tilde{D}](1 - e^{-2\gamma}) \right\} \times \exp \left\{ -\frac{\gamma t}{2} \right\} \chi(\beta,0), \]

where \( \chi(\beta,0) \) is the initial-time CF of the two modes and we have assumed that \( M \) is real.

For convenience, we assume that the two modes are prepared initially in a more general two-mode Gaussian state with the covariance matrix of the form [24]

\[ \mathbf{V} = \begin{pmatrix} n_1 & 2m_1 & m_s & m_c \\ 2m_1 & n_1 & m_s^* & m_c^* \\ m_s^* & m_c^* & n_2 & 2m_2 \\ m_c^* & m_s^* & 2m_2 & n_2 \end{pmatrix}. \]

In order to be a physically acceptable state [25], \( \mathbf{V} \) must be positive semidefinite and satisfies the uncertainty principle \( \mathbf{V} + \mathbf{I}/2 \geq 0 \). Here \( \mathbf{I} \) is the standard symplectic form

\[ \mathbf{I} : = \mathbf{J} \oplus \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Finally, a general solution of Eq. (9) subjected to the initial condition (10) is given by

\[ \gamma_{SSVS} = \gamma^I \oplus \gamma^S = \frac{1}{2} \begin{pmatrix} \cosh 2r & \sinh 2r & 0 & 0 \\ \sinh 2r & \cosh 2r & 0 & 0 \\ 0 & 0 & \cosh 2r & \sinh 2r \\ 0 & 0 & \sinh 2r & \cosh 2r \end{pmatrix}, \]

where we have assumed that the two modes are equally squeezed and \( r \) is the squeezing constant.

Substituting Eq. (13) into Eq. (12), we can easily obtain the time-evolved characteristic function for the two modes as

\[ \chi(\beta,t) = \exp \left\{ -\frac{1}{4} \mathbf{\Gamma}_S(t)_{SSVS} \mathbf{\Lambda} \right\} \]

where \( \mathbf{\Gamma}_S = \gamma^I \oplus \gamma^S \).
where \( \Lambda=(\beta_1^0,\beta_1,\beta_2^0,\beta_2)^T \) and the entries of the matrix are given by

\[
\begin{align*}
S_1(t) &= \left( N + \frac{1}{2} \right) \tau - \frac{1}{2} \cosh(2r)(\tau - 2), \\
S_2(t) &= M \tau - \frac{1}{2} \sinh(2r)(\tau - 2), \\
S_C(t) &= \left[ N + \frac{1}{2} - \frac{1}{2} \cosh(2r) \right] \tau, \\
S_D(t) &= \left[ M - \frac{1}{2} \sinh(2r) \right] \tau.
\end{align*}
\]

We introduce a \( 4 \times 4 \) matrix

\[
F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \oplus \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},
\]

so that the covariance matrix (14) can be transformed into the following form:

\[
\Gamma_{SSVS}^{\prime}(t) = \frac{1}{2} \begin{pmatrix} n_1 & 0 & c_1 & 0 \\ 0 & n_2 & 0 & c_2 \\ c_1 & 0 & n_1 & 0 \\ 0 & c_2 & 0 & n_2 \end{pmatrix},
\]

with

\[
\begin{align*}
n_1 &= \frac{1}{2} \left[ (\tilde{N} - e^{2r} + 2M) \tau + 2e^{2r} \right], \\
n_2 &= \frac{1}{2} \left[ (\tilde{N} - e^{-2r} + 2M) \tau + 2e^{-2r} \right], \\
c_1 &= \frac{1}{2} \left[ \tilde{N} + 2M - e^{2r} \right] \tau, \\
c_2 &= \frac{1}{2} \left[ \tilde{N} + 2M - e^{-2r} \right] \tau,
\end{align*}
\]

where \( \tilde{N}=2N+1 \).

It is well known that the correlation matrix of an arbitrary bipartite two-mode system can be written in block form as

\[
V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},
\]

where \( A \) and \( B \) are \( 2 \times 2 \) Hermitian matrices containing only local elements, while \( C \) is the correlation between two subsystems. One has found that Simon’s separability criterion is the most successfully CV version of the Peres-Horodecki partial-transpose criterion, because it is not only a necessary but also a sufficient condition for the separability of any two-mode Gaussian state [26]. It reads as follows:

\[
\Delta_s = \det A \det B + \left( \frac{1}{4} - |\det C| \right)^2 - \text{Tr}[A^C J B^C J],
\]

\[
- \frac{1}{4} (\det A + \det B),
\]

where the matrix \( J \) is given by Eq. (11). He has shown that when \( \Delta_s \geq 0 \), then the state is separable, otherwise it is entangled.

For a Gaussian state (17), the Simon criterion can be rewritten as [19]

\[
\Delta_s = [(n_1 - |c_1|)(n_2 - |c_2|) - 1][(n_1 + |c_1|)(n_2 + |c_2|) - 1] \geq 0.
\]

Substituting Eq. (18) into inequality (21), we can obtain

\[
\Delta_s = (\tilde{N} - 2M - e^{-2r})(\tilde{N} + 2M - e^{2r}) \geq 0.
\]

One sees that determining whether this inequality is violated is not an easy task. So, we will consider respectively two types of environment: a heat bath and a squeezed environment. For the former (\( M=0 \)), this inequality is violated only when the squeezing degree, \(|r|\), of the initial state is greater than the squeezing parameter, \( r_{eS} \), of the thermal field, i.e.,

\[
|r| > r_{eS} \equiv \frac{1}{2} \ln(2N + 1).
\]

In this case, such a chaotic field can entangle two modes, in which they are prepared initially in their single-mode squeezed states; for the latter, i.e., \( M=\sqrt{N(N+1)} \), we find that the Simon criterion is equal to zero if and only if the initial squeezing parameter \( |r| \) of the quantum state is equal to that of such an environment, namely

\[
|r| = r_{eS} \equiv \frac{1}{2} \text{arcsinh}(2M),
\]

except that the Simon criterion is always negative, showing that such a noisy field can induce entanglement between the two modes that are initially both in their squeezed states as long as \(|r| \neq r_{eS} \).

In the following, we will consider carefully the entanglement dynamics of the two modes in the influence of two kinds of environment mentioned above. Before doing this, we should point out that our decoherent model can be viewed as a Gaussian noise channel, so the evolved state is Gaussian. Also, we have treated the interaction of a quantum system with the external environment as the Markovian and rotating-wave approximations; the considered environment cannot lead to blatantly nonphysical behavior of any Gaussian state in the following investigation.

We use the logarithmic negativity to quantify the amount of entanglement for the two modes. The logarithmic negativity is defined as [27]
Obviously, the state is separable for $\eta^{-} = \frac{1}{2}$. We first plot the dynamics of the logarithmic negativity of the two modes in a common thermal environment in Eq. (23). In addition, as can be seen from Fig. 1, the common vacuum field seems always to be better at generating and enhancing entanglement between the two uncorrelated modes than the common thermal environment.

Next, we plot the dynamical behavior of the logarithmic negativity of the two modes embedded in a common squeezed vacuum environment, as shown in Fig. 2. Similar to the above case, the two uncorrelated modes can be entangled from an initial separable state including a vacuum state by simultaneously interacting with the squeezed vacuum environment. However, the generated entanglement can exhibit different behaviors depending on the squeezing parameters of such an environment and of the initial state. To gain some insights, we define a quantity $\Delta_N = |r - r_{s5}|$, which is the absolute value of the squeezing parameter of a quantum state differing from that of the squeezed bath. Thus, based on Eq. (24) and the parameters given by Fig. 2(a), we have $\Delta_N = 0.14436$. Furthermore, it is not difficult to see that the larger $\Delta_N$ can help to generate and enhance the entanglement at a later time. In other words, the closer the initial squeezing parameter of the two modes is to $r_{s5}$, the harder it is to generate entanglement. To see this, we plot the logarithmic negativity as a function of the initial squeezing parameter at different interaction time $\tau$ in Fig. 2(b). We find that at $|\tau|=1.4436$, which is equal to the squeezing parameter of the reservoir with $M=2\sqrt{5}$, the two modes cannot be entangled forever at later time even though they are coupled to a common squeezed bath, i.e., their entanglement $E_N=0$. A natural question arises from Figs. 1(b) and 2(b): Could an entangled Gaussian bath leave its initial entanglement invariant under the influence of the common noisy environment if its squeezing parameter is equal to that of the ambient reservoir? This sets the stage for us to study next the entanglement dynamics of a two-mode entangled Gaussian state in a common environment.

It is well known that since any two-mode entangled Gaussian state is locally equivalent to a two-mode squeezed

\[ E_N := \max \{0, -\log_2(2\eta^-)\}, \] (25)

where $\eta^-$ is the smallest symplectic eigenvalue of the partially transposed covariance of two-mode Gaussian state (19) and given by

\[ (\eta^-)^2 = \frac{1}{2} [\det A + \det B - 2 \det C - \sqrt{(\det A + \det B - 2 \det C)^2 - 4 \det V}], \] (26)

\[ \log_{2} 2 = 1 - \exp(-2\gamma t). \]
states and coupled to the same squeezed vacuum bath.

Following the same procedure as before, it is easy to arrive at the following covariance matrix at time \( r \) as

\[
\mathbf{V}_{\text{T SvS}} = \frac{1}{2} \begin{pmatrix}
\cosh(2r) & 0 & 0 & \sinh(2r) \\
0 & \cosh(2r) & \sinh(2r) & 0 \\
0 & \sinh(2r) & \cosh(2r) & 0 \\
\sinh(2r) & 0 & 0 & \cosh(2r)
\end{pmatrix},
\]

where \( r \) is the degree of two-mode squeezing.

Following the same procedure as before, it is easy to arrive at the following covariance matrix at time \( t \) as

\[
\Gamma_{\text{T SvS}}(t) = \frac{1}{2} \begin{pmatrix}
n_{T1} & 0 & c_{T1} & 0 \\
0 & n_{T2} & 0 & c_{T2} \\
c_{T1} & 0 & n_{T1} & 0 \\
0 & c_{T2} & 0 & n_{T2}
\end{pmatrix},
\]

with

\[
\begin{align*}
n_{T1} &= \frac{1}{2}(\tilde{N} - e^{-2r} + 2M)\tau + \cosh(2r), \\
n_{T2} &= \frac{1}{2}(\tilde{N} - e^{-2r} - 2M)\tau + \cosh(2r), \\
c_{T1} &= \frac{1}{2}(\tilde{N} - e^{-2r} + 2M)\tau + \sinh(2r), \\
c_{T2} &= \frac{1}{2}(\tilde{N} - e^{-2r} - 2M)\tau - \sinh(2r)
\end{align*}
\]

where, as before, \( \tilde{N} = 2N + 1 \). We see that if the two modes are embedded in a thermal environment, i.e., \( (M=0) \), then Eq. (28) reduces to the covariance matrix derived by a Fokker-Planck equation for the Wigner function of the system in Ref. [19], showing that our method and that of Ref. [19] are equivalent to each other in describing the entanglement dynamics of quantum continuous variable systems under the influence of a noisy environment. Nevertheless, the advantage of our method is that one can easily obtain the exact solution to the Fokker-Planck equation for the characteristic function of many harmonic oscillators embedded in a common reservoir by introducing some operators similar to Eq. (7).

Substituting Eq. (29) into inequality (21), we can obtain the Simon criterion as

\[
\Delta_s = e^{-2r} + (\tilde{N} - e^{-2r} - 2M)e^{-2r}\tau - 1 \geq 0.
\]

We see that for a case of \( N > M \), if the initially squeezing parameter of the two-mode squeezed vacuum state satisfies

\[
|r| > \frac{1}{2} \ln[2(N - M) + 1],
\]

the inequality is always violated at a later time, implying that it will remain entangled forever in spite of the interaction with the noisy environment. If \( |r| \leq \frac{1}{2} \ln[2(N - M) + 1] \), then the state becomes disentangled after a transmission time,

\[
t = \frac{1}{2\gamma} \ln \left( \frac{\tilde{N} - 2M - e^{-2|r|}}{\tilde{N} - 2M - e^{2|r|}} \right).
\]

But for a case of \( N < M < \sqrt{N(N+1)} \), we find that \( \Delta_s \) is always a negative value, showing that any two-mode squeezed vacuum state is the disentangled-free state in such an environment.

A more detailed analysis on the entanglement evolution of two-mode squeezed vacuum states embedded in a common thermal environment has been given in Ref. [19]. We do not repeat it here, but emphasize that such a thermal bath gives
rise to the initial entanglement degradation of two-mode squeezed vacuum states. So, we return our attention to the investigation of the entanglement dynamics of a two-mode squeezed vacuum state in a common squeezed environment.

We can obtain the entanglement evolution through the computation of the logarithmic negativity using Eqs. (25), (26), and (28). Figure 3(a) shows the time evolution of entanglement of the two-mode squeezed vacuum state embedded in various squeezed baths, whereas Fig. 3(b) displays the time evolution of entanglement of the two-mode squeezed vacuum state with different squeezing parameters interacting with the same squeezed bath with \( N = 4 \) and \( M = 2\sqrt{5} \). From Fig. 3 we can see that whether the entanglement enhances or reduces is determined strongly by the squeezing parameter of such a state and of the environment. We find that the squeezed bath has the intrinsic squeezing parameter given by \( r_{\text{eff}} = \frac{1}{2} \arcsinh(2M) \). Thus, in Fig. 3(a) all cases \( N = (0.2, 0.5, 1, 1.5, 2, 4) \); we can compute the corresponding squeezing parameters of the baths as \((0.4335, 0.6585, 0.8814, 1.0317, 1.1462, 1.4436)\). Furthermore, we can easily arrive at a general conclusion: If the squeezing parameter of a quantum state is larger than that of the squeezed environment, i.e., \( r > r_{\text{eff}} \), then the entanglement decreases initially with time. The larger the values of \( r - r_{\text{eff}} \) are, the faster the entanglement can be degraded. Especially, for a squeezed bath with \( N > M \), we see that when the initial squeezing parameter of a two-mode squeezed vacuum state is below \( \frac{1}{2} \ln [2(N-M)+1] \), the entanglement is completely lost in a finite time. On the other hand, if the squeezing parameter of a quantum state is less than that of the environment, i.e., \( r < r_{\text{eff}} \), the entanglement grows initially with time. The larger the values of \( r_{\text{eff}} - r \) are, the faster the entanglement that can be increased. The degree of entanglement is maximized at the limit of \( \tau = 1 \). Thus, our investigation shows that such a squeezed bath has dual characters: constructive and destructive roles. But if we know the structure and type of the ambient environment beforehand, we can exploit it as the third party to enhance the initial entanglement of two-mode CV Gaussian states owned by the two users. Indeed, schemes for engineering the form of the system-reservoir coupling and the state of the reservoir have been proposed theoretically and experimentally in recent years [28–30]. For example, in Ref. [28], the authors showed that a squeezed environment can be obtained by means by feedback schemes relying on quantum nondemolition “intracavity” measurement. Myaat et al. [30] applied noisy potentials to the trap electrodes to simulate a hot reservoir so that the system of a trapped ion can have a controllable reservoir. Therefore, we hope that this interesting phenomenon is demonstrated within current technology and can be used for quantum-information processing in the near future. Finally, from Fig. 3 we also see an interesting phenomenon, that is, when \( |r| = r_{\text{eff}} \), the two-mode squeezed vacuum state can not be affected at all by a given squeezed bath, showing that such an initial two-mode squeezed vacuum state is a decoherence-free state in a given squeezed bath. In fact, this condition for a two-mode squeezed vacuum state to be DFS can be derived from Eq. (29). When the expressions

\[
\tilde{N} - e^{2r} + 2M = 0 \quad \text{and} \quad \tilde{N} - e^{-2r} - 2M = 0
\]

are satisfied, we can obtain \( |r| = r_{\text{eff}} = \frac{1}{2} \arcsinh(2M) \). That is to say, under this condition, the evolving covariance matrix is always equal to the covariance matrix of the two-mode squeezed vacuum state in spite of the interaction between the system and the external environment. In addition, since any two-mode Gaussian entangled states are locally equivalent to the two-mode squeezed vacuum state of suitable squeezing, this condition holds for all two-mode entangled Gaussian states in the common squeezed bath.

IV. SUMMARY

To summarize, we have derived an exact master equation for the two noninteracting modes embedded in a common environment and obtained a general solution associated with
Interestingly, we have found a condition for any two-mode Gaussian state by means of the characteristic function method. As a concrete example, we have investigated the entanglement dynamics of the two modes in various initial states—a separable state and two-mode squeezed vacuum state—embedded in different environments. It is found that only when their initial squeezing parameter is larger than that of the thermal bath can the two modes get entanglement from the initial separable states. As for the common squeezed bath, there is no entanglement between the two modes only in a case of $|\alpha|^2 = |\alpha_0|^2$. Once created, entanglement is enhanced by the external bath at a later time.

On the other hand, for a two-mode squeezed vacuum state, the entanglement dynamics between the two modes is determined strongly by the difference between their initial squeezing parameter and the environment’s squeezing parameter. If the difference is greater than zero, then the entanglement between two modes reduces. If the difference is less than zero, then the entanglement between two modes increases. Interestingly, we have found a condition for any two-mode Gaussian state to be DFS in a given squeezed bath, that is, the difference is equal to zero.

Of course, the interaction between the quantum system and its surroundings is not easy to understand. Our results are shown to be valid in the case of a noisy Gaussian channel. Other noisy models have yet to be investigated. However, it is hoped that the present research has shed some light on the subject.

ACKNOWLEDGMENTS

This work was funded by the National Natural Science Foundation of China under Grant No. 10374007, the Natural Science Foundation of Hunan Province under Grant No. 06JJ500143, the Natural Science Foundation of the Education Department of Hunan Province under Grant No. 05C696, and the Young Core Teachers Foundation of Hunan Provincial Education Department.


